1702EC402–Signals and Systems

Academic Year :	**2018-2019**	**Question Bank**	Programme:	B.E – ECE
Year / Semester :	*II / IV*		Course Coordinator:	R.keerthika

PART – A (2 Mark Questions With Key)				
S.No	**Questions**	**Mark**	**COs**	**BTL**
UNIT IV –ANALYSIS OF DISCRETE TIME SIGNALS				
1	**What is meant by step response of the DT system?** The output of the system y(n) is obtained for the unit step input u(n) then it is said to be step response of the system.	2	4	K1
2	**Define Transfer function of the DT system.** The Transfer function of DT system is defined as the ratio of Z transform of the system output to the input. That is ,H(z)=Y(z)/X(z).	2	4	K1
3	**Define impulse response of a DT system. (APRIL/MAY 2011)** The impulse response is the output produced by DT system when unit impulse is applied at the input. The impulse response is denoted by h(n) The impulse response h(n) is obtained by taking inverse Z transform from the transfer function H(z).	2	4	K1
4	**State the significance of difference equations** The input and output behavior of the DT system can be characterized with the help of linear constant coefficient difference equations.	2	4	K1
5	**Write the difference equation for Discrete time system.** The general form of constant coefficient difference equation is Y(n) = -Σ ak y(n-k) +Σ bk x(n-k) Here n is the order of difference equation.x(n) is the input and y(n) is the output.	2	4	K1
6	**Define frequency response of the DT system.** The frequency response of the system is obtained from the Transfer function by replacing z = ejωIe, H(z)=Y(z)/X(z), Where z = ejω	2	4	K1
7	**What is the condition for stable system?** A LTI system is stable if Σ¦h(n)¦< ∞ . Here the summation is absolutely summable	2	4	K1

8	**What are the blocks used for block diagram representation?**	2	4	
	The block diagrams are implemented with the help of scalar multipliers, adders and multipliers			K1
9	**State the significance of block diagram representation**	2	4	
	The LTI systems are represented with the help of block diagrams. The block diagrams are more effective way of system description. Block Diagrams Indicate how individual calculations are performed. Various blocks are used for block diagram representation			K1
10	**What are the properties of convolution?**	2	4	
	i.Commutative ii.Assosiative. iii.Distributive			K1
11	**State the Commutative properties of convolution?**	2	4	
	Commutative property of Convolution is x(t)*h(t)=h(t)*x(t)			K1
12	**State the Associative properties of convolution**	2	4	
	Associative Property of convolution is [x(t)*h1(t)]*h2(t)=x(t)*[h1(t)*h2(t)]			K1
13	**State Distributive properties of convolution**	2	4	
	The Distributive Property of convolution is {x(t)*[h1(t)+ h2(t)]}= x(t)*h1(t) + x(t)*h2(t)]			K1
14	**Define causal system**	2	4	
	For a LTI system to be causal if h(n)=0,for n<0.			K1
15	**What is the impulse response of the system y(t)=x(t-t0).**	2	4	
	h(t)=δ(t-t0)			K1

PART – B (12 Mark Questions with Key)

S.No	Questions	Mark	CO	BTL
1	**I0Find the z transform of the signal x(n)=[$sin\omega o$n]u(n)** **ii)determine the z transform ,roc,and pole zero location of x(z) for** **x(n)=$(\frac{2}{3})^n$ u(n)+$(\frac{-1}{2})^n$ u(n)**	**12**	**4**	**K2**

1 ii)

Solved Problem 10.9 Determine the z-transform, ROC and pole-zero locations of $X(z)$ for

$$x(n) = \left(\frac{2}{3}\right)^n u(n) + \left(-\frac{1}{2}\right)^n u(n)$$

Solution:

Given $x(n) = \left(\frac{2}{3}\right)^n u(n) + \left(-\frac{1}{2}\right)^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\left(\frac{2}{3}\right)^n u(n) + \left(-\frac{1}{2}\right)^n u(n)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{2}{3} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{2} z^{-1}\right)^n$$

The first summation converges for $\left|\frac{2}{3} z^{-1}\right| < 1$ or $|z| > \frac{2}{3}$ and the second summation converges for $\left|-\frac{1}{2} z^{-1}\right| < 1$ or $|z| > \frac{1}{2}$. Therefore $X(z)$ converges for $|z| > \frac{2}{3}$.

Now

$$X(z) = \frac{1}{1 - \frac{2}{3} z^{-1}} + \frac{1}{1 + \frac{1}{2} z^{-1}} \qquad \left(\because\ 1 + r + r^2 + \cdots + \infty = \frac{1}{1-r}\right)$$

$$= \frac{z}{z - \frac{2}{3}} + \frac{z}{z + \frac{1}{2}}$$

$$= \frac{2z^2 - \frac{1}{6}z}{\left(z - \frac{2}{3}\right)\left(z + \frac{1}{2}\right)}$$

$$= \frac{z\left(2z - \frac{1}{6}\right)}{\left(z - \frac{2}{3}\right)\left(z + \frac{1}{2}\right)} \text{ ROC ; } |z| > \frac{2}{3} \qquad (10.25)$$

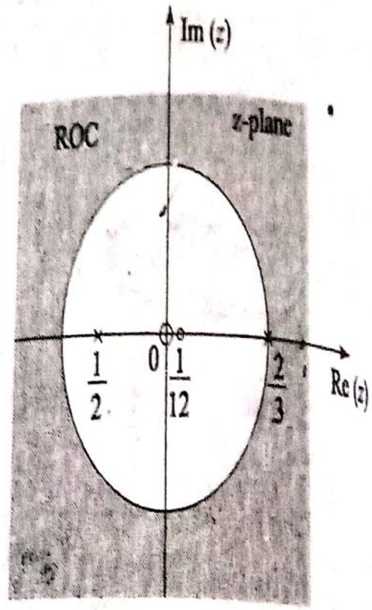


Fig.10.6

The poles of $X(z)$ are at $z = \frac{2}{3}$ and at $z = -\frac{1}{2}$ and the zeros of $X(z)$ are at $z = 0$ and at $z = \frac{1}{12}$.

	Solved Problem 10.8 Find the z-transform of the signal $x(n) = (\sin\omega_o n)u(n)$ and find ROC. ***Solution:*** Given $x(n) = (\sin\omega_o n)u(n)$ $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$ $= \sum_{n=-\infty}^{\infty} (\sin\omega_o n)u(n)z^{-n} = \sum_{n=0}^{\infty}(\sin\omega_o n)z^{-n}$ $= \sum_{n=0}^{\infty}\left[\frac{e^{j\omega_o n} - e^{-j\omega_o n}}{2j}\right]z^{-n}$ $\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$ $= \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{j\omega_o n} - e^{-j\omega_o n}\right)z^{-n}$ $= \frac{1}{2j}\left[\sum_{n=0}^{\infty} e^{j\omega_o n}z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega_o n}z^{-n}\right]$ $= \frac{1}{2j}\left[\sum_{n=0}^{\infty}\left(e^{j\omega_o}z^{-1}\right)^n - \sum_{n=0}^{\infty}\left(e^{-j\omega_o}z^{-1}\right)^n\right]$ If $\lvert e^{j\omega_o}z^{-1}\rvert < 1$ and $\lvert e^{-j\omega_o}z^{-1}\rvert < 1$, the above series conv That is for $\lvert z\rvert > 1$, the series converges and we get Discrete-Time Signal and System Analysis using the z-Transform 727 $X(z) = \frac{1}{2j}\left[\frac{1}{1-e^{j\omega_o}z^{-1}} - \frac{1}{1-e^{-j\omega_o}z^{-1}}\right]$ $\boxed{\sum_{n=0}^{\infty} = 1 + r + r^2 + \cdots + \infty = \frac{1}{1-r} \text{ if } \lvert r\rvert < 1}$ $= \frac{1}{2j}\left[\frac{1 - e^{-j\omega_o}z^{-1} - 1 + e^{j\omega_o}z^{-1}}{(1-e^{j\omega_o}z^{-1})(1-e^{-j\omega_o}z^{-1})}\right]$ $= \frac{1}{2j}\left[\frac{e^{j\omega_o}z^{-1} - e^{-j\omega_o}z^{-1}}{1 - e^{j\omega_o}z^{-1} - e^{-j\omega_o}z^{-1} + z^{-2}}\right]$ $= \frac{(\sin\omega_o)z^{-1}}{1 - 2(\cos\omega_o)z^{-1} + z^{-2}}$. ROC : $\lvert z\rvert > 1$ (10.24)			
2	**Using long division method determine the inverse of z transform** **Of x(z)= $1+2z^{-1}\backslash 1-2z^{-1}+z^{-2}$**	**12**	**4**	**K2**

Solved Problem 10.22 Using long division, determine the inverse z-transform of

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

if (a) $x(n)$ is causal and (b) is anticausal.

Solution:

(a) Given $X(z) = \dfrac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$

$$
\begin{array}{r|l}
 & 1+4z^{-1}+7z^{-2}+10z^{-3}+13z^{-4}+16z^{-5} \\
\hline
1-2z^{-1}+z^{-2} & 1+2z^{-1} \\
 & 1-2z^{-1}+z^{-2} \\
 & \quad 4z^{-1}-z^{-2} \\
 & \quad 4z^{-1}-8z^{-2}+4z^{-3} \\
 & \qquad 7z^{-2}-4z^{-3} \\
 & \qquad 7z^{-2}-14z^{-3}+7z^{-4} \\
 & \qquad\quad 10z^{-3}-7z^{-4} \\
 & \qquad\quad 10z^{-3}-20z^{-4}+10z^{-5} \\
 & \qquad\qquad 13z^{-4}-10z^{-5} \\
 & \qquad\qquad 13z^{-4}-26z^{-5}+13z^{-6} \\
 & \qquad\qquad\quad 16z^{-5}-13z^{-6} \\
 & \qquad\qquad\quad 16z^{-5}-32z^{-6}+16z^{-7} \\
 & \qquad\qquad\qquad 19z^{-6}-16z^{-7}
\end{array}
$$

The sequence $x(n)$ is causal and $X(z)$ is expressed in negative powers of z. Therefore we begin the division with a constant.

$\therefore X(z) = 1+4z^{-1}+7z^{-2}+10z^{-3}+13z^{-4}+16z^{-5}+\cdots$

$x(n) = \{\underset{\uparrow}{1}, 4, 7, 10, 13, 16, 19, \ldots\}$

(b)

$$
\begin{array}{l|l}
 & 2z+5z^2+8z^3+11z^4+14z^5\ldots\ldots \\
z^{-2}-2z^{-1}+1 & 2z^{-1}+1 \\
 & 2z^{-1}-4+2z \\
 & \quad 5-2z \\
 & \quad 5-10z+5z^2 \\
 & \quad\quad 8z-5z^2 \\
 & \quad\quad 8z-16z^2+8z^3 \\
 & \quad\quad\quad 11z^2-8z^3 \\
 & \quad\quad\quad 11z^2-22z^3+11z^4 \\
 & \quad\quad\quad\quad 14z^3-11z^4 \\
 & \quad\quad\quad\quad 14z^3-28z^4+14z^5 \\
 & \quad\quad\quad\quad\quad 17z^4-14z^5
\end{array}
$$

Since $x(n)$ is non-causal we begin the division with highest negative of z.

$$X(z)=2z+5z^2+8z^3+11z^4+14z^5+\ldots$$

$$x(n)=\{\ldots,14,11,8,5,2,\underset{\uparrow}{0}\}$$

3	**i)find the inverse of z transform of the following** $x(z)=\frac{1}{4}z^{-1}\backslash(1--\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})$ROC:$z>\frac{1}{2}$ **ii)explain the properties of ROC?**	12	4	K2

i) Given

$$X(z) = \frac{\frac{1}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)};\ |z| > \frac{1}{2}.$$

On multiplying numerator and denominator with z^2, we get

$$\frac{X(z)}{z} = \frac{\frac{1}{4}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

The above equation can be expressed in partial fraction as follows

$$\frac{X(z)}{z} = \frac{c_1}{z-\frac{1}{2}} + \frac{c_2}{z-\frac{1}{4}}$$

where c_1 and c_2 can be evaluated using Eq. (10.74).

That is

$$c_1 = \left(z-\frac{1}{2}\right)\frac{\frac{1}{4}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\Bigg|_{z=\frac{1}{2}} = 1$$

$$c_2 = \left(z-\frac{1}{4}\right)\frac{\frac{1}{4}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\Bigg|_{z=\frac{1}{4}} = -1$$

$$\frac{X(z)}{z} = \frac{1}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{4}}$$

$$X(z) = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}} \quad \text{ROC: } |z| > \frac{1}{2}$$

Since ROC is $|z| > \frac{1}{2}$, the sequence is causal and using table (10
write

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$$

4	**Determine all possible signals x(n) associated with z transform $x(z)=5z^{-1}/(1-2z^{-1})(1-3z^{-1})$**	12	4	K2

Multiplying numerator and denominator with z^2, we obtain

$$X(z) = \frac{5z}{(z-2)(z-3)}$$

$$\frac{X(z)}{z} = \frac{5}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$A = (z-2)\frac{5}{(z-2)(z-3)}\bigg|_{z=2} = -5$$

$$B = (z-3)\frac{5}{(z-2)(z-3)}\bigg|_{z=3} = 5$$

$$X(z) = \frac{-5z}{z-2} + \frac{5z}{z-3} \qquad (10.86)$$

The pole-zero plot of $X(z)$ is shown in Fig. 10.7.

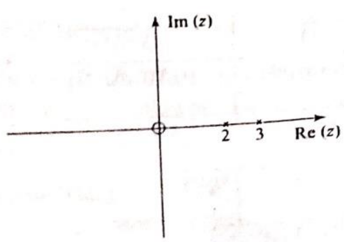


Fig. 10.7

For the ROC: $|z| > 3$ shown in Fig. 10.8, the signal $x(n)$ is causal and all terms in Eq. (10.86) are causal terms.

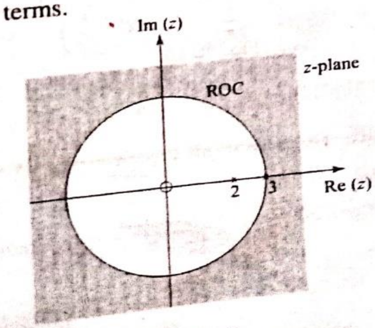


Fig. 10.8

$$x(n) = -5(2)^n u(n) + 5(3)^n u(n)$$

For ROC: $|z| < 2$, shown in Fig. 10.9, the signal $x(n)$ is anticausal and all terms in Eq. (10.86) are anticausal terms.

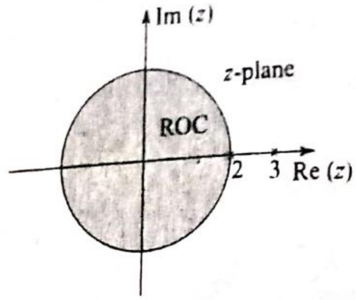


Fig. 10.9

Therefore $$x(n) = 5(2)^n u(-n-1) - 5(3)^n u(-n-1)$$

For ROC $2 < |z| < 3$, shown in Fig. 10.10 the signal $x(n)$ is two sided. The pole $z = 2$ provides causal term and the pole $z = 3$ provides the noncausal term. Therefore

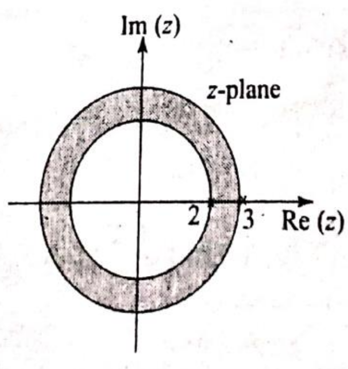


Fig. 10.10

$$x(n) = -5(2)^n u(n) - 5(3)^n u(-n-1)$$

5	**i)find the fouriertransformof x(n)=sin($\frac{\pi n}{2}$)u(n)** **ii)find the DTFT of the following x(n)=$(0.5)^n$ u(n)+2^{-n}u(-n-1)**	**12**	**4**	**K2**

Solved Problem 9.13 Find the Fourier transform of $x(n) = \sin\left(\frac{\pi n}{2}\right) u(n)$

Given

$$x(n) = \sin\left(\frac{\pi n}{2}\right) u(n)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{\frac{j\pi n}{2}} - e^{\frac{-j\pi n}{2}}}{2j}\right] e^{-j\omega n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} e^{\frac{j\pi n}{2}} e^{-j\omega n} - \sum_{n=0}^{\infty} e^{\frac{-j\pi n}{2}} e^{-j\omega n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} e^{\left(\frac{j\pi}{2} - j\omega\right)n} - \sum_{n=0}^{\infty} e^{\left(\frac{-j\pi}{2} - j\omega\right)n}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1 - e^{j\pi/2} e^{-j\omega}} - \frac{1}{1 - e^{-j\pi/2} e^{-j\omega}}\right]$$

$$= \frac{1}{2j}\left[\frac{1 - e^{-j\pi/2} e^{-j\omega} - 1 + e^{j\pi/2} e^{-j\omega}}{1 - e^{-j\omega}(e^{j\pi/2} + e^{-j\pi/2}) + e^{-j2\omega}}\right]$$

$$= \frac{1}{2j}\left[\frac{e^{-j\omega}(e^{j\pi/2} - e^{-j\pi/2})}{1 - 2\cos\frac{\pi}{2} e^{-j\omega} + e^{-j2\omega}}\right]$$

$$= \frac{e^{-j\omega} \sin\frac{\pi}{2}}{1 + e^{-j2\omega}}$$

$$= \frac{e^{-j\omega}}{1 + e^{-j2\omega}}$$

	Discrete-Time Signal and System Analysis us $= \sum_{n=0}^{\infty}(0.5)^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} 2^n e^{-j\omega n}$ $= \sum_{n=0}^{\infty}(0.5e^{-j\omega})^n + \sum_{n=1}^{\infty}(2^{-1}e^{j\omega})^n$ $= \frac{1}{1-0.5e^{-j\omega}} + \frac{0.5e^{j\omega}}{1-0.5e^{j\omega}}$ $= \frac{1-0.5e^{j\omega}+0.5e^{j\omega}-0.25}{1-\cos\omega+0.25}$ $= \frac{0.75}{1.25-\cos\omega}$ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty}(0.5)^n u(n)e^{-j\omega n} + \sum_{n=-\infty}^{\infty} 2^n u(-n-1)e^{-j\omega}$			
6	**State and prove the properties of DTFT**	**12**	**4**	**K2**

(9.43)

Time reversal

If $\mathscr{F}[x(n)] = X(e^{j\omega})$

then

$$\mathscr{F}[x(-n)] = X(e^{-j\omega}) \quad (9.44)$$

Proof

$$\mathscr{F}[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} x(n)e^{j\omega n} = \sum_{n=-\infty}^{\infty} x(n)e^{-(-j\omega)n}$$
$$= X(e^{-j\omega})$$

That is folding in the time domain corresponds to the folding in the frequency domain.

Differentiation in frequency

If $\mathscr{F}[x(n)] = X(e^{j\omega})$

then

$$\mathscr{F}[nx(n)] = \frac{jd}{d\omega}X(e^{j\omega}) \quad (9.45)$$

Proof

$$X(e^{j\omega}) = \mathscr{F}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Frequency shifting property

If $\mathscr{F}[x(n)] = X(e^{j\omega})$

$$\mathscr{F}\left[x(n)e^{j\omega_0 n}\right] = X\left[e^{j(\omega-\omega_0)}\right]$$

This property is the dual of the time - shifting property.

Periodicity

The discrete-time Fourier transform $X(e^{j\omega})$ is periodic in ω with period 2π, satisfying the following condition

$$X(e^{j\omega}) = X\left[e^{j(\omega+2k\pi)}\right] \tag{9.40}$$

for any integer k.

Implication: We need only one period of $X(e^{j\omega})$ (i. e., $\omega \in [0, 2\pi]$ or $[-\pi, \pi]$) for analysis and not the whole range $-\infty < \omega < \infty$.

Time shifting

$$\text{If} \quad \mathscr{F}[x(n)] = X(e^{j\omega})$$

then

$$\mathscr{F}[x(n-k)] = e^{-j\omega k} X\left(e^{j\omega}\right) \quad \text{where } k \text{ is an integer.} \tag{9.41}$$

Proof

$$\begin{aligned}
\mathscr{F}[x(n-k)] &= \sum_{n=-\infty}^{\infty} x(n-k)e^{-j\omega n} \\
&= \sum_{P=-\infty}^{\infty} x(P)e^{-j\omega(P+k)} \qquad \text{put } n-k=P \Rightarrow n = P+k \\
&= e^{-j\omega k}\sum_{P=-\infty}^{\infty} x(P)e^{-j\omega P} \\
&= e^{-j\omega k} X\left(e^{j\omega}\right)
\end{aligned}$$

The above result shows that time shifting of a signal by k units does not change its amplitude spectrum. The phase spectrum is changed by $-\omega k$.

9.3.2 Properties of discrete-time Fourier transform

In this section, we will study some of the important properties of discrete-time Fourier transform.

Linearity

If

$$\mathcal{F}\,[x_1(n)] = X_1(e^{j\omega}) \tag{9.37}$$

and

$$\mathcal{F}\,[x_2(n)] = X_2(e^{j\omega}) \tag{9.38}$$

then

$$\mathcal{F}\,[a_1x_1(n) + a_2x_2(n)] = a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega}) \tag{9.39}$$

Proof

$$\begin{aligned}\mathcal{F}\,[a_1x_1(n) + a_2x_2(n)] &= a_1\mathcal{F}\,[x_1(n)] + a_2\mathcal{F}\,[x_2(n)]\\ &= a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega})\end{aligned}$$

The correlation theorem

If

$$\mathcal{F}\,[x_1(n)] = X_1(e^{j\omega})$$

and

$$\mathcal{F}\,[x_2(n)] = X_2(e^{j\omega})$$

then

$$\mathcal{F}\,\left[\gamma_{x_1x_2}(\ell)\right] = \Gamma_{x_1x_2}\left(e^{j\omega}\right) = X_1\left(e^{j\omega}\right)\, X_2\left(e^{-j\omega}\right) \tag{9.49}$$

The function $\Gamma_{x_1x_2}\left(e^{j\omega}\right)$ is called the cross energy density spectrum of the signals $x_1(n)$ and $x_2(n)$.

That is the convolution of two signals in time domain is equal to multiplying their spectra in the frequency domain.

Convolution in frequency domain

This is a dual of the convolution property

If $\mathcal{F}[x_1(n)] = X_1(e^{j\omega})$

and $\mathcal{F}[x_2(n)] = X_2(e^{j\omega})$

then

$$\mathcal{F}[x_1(n)x_2(n)] = X_1(e^{j\omega}) * X_2(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta \qquad (9.48)$$

Differentiate both sides with respect to ω

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} (-jn)x(n)\, e^{-j\omega n}$$

$$= (-j) \sum_{n=-\infty}^{\infty} n\, x(n)\, e^{-j\omega n}$$

$$\Rightarrow j\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} n\, x(n)\, e^{-j\omega n}$$

$$= \mathcal{F}\,[nx(n)]$$

Therefore $\mathcal{F}\,[nx(n)] = j\dfrac{dX(e^{j\omega})}{d\omega}$

Convolution in time domain

If $\mathcal{F}\,[x_1(n)] = X_1(e^{j\omega})$

and

$\mathcal{F}\,[x_2(n)] = X_2(e^{j\omega})$

then

$$\mathcal{F}\,[x_1(n) * x_2(n)] = X_1(e^{j\omega})\, X_2(e^{j\omega})$$

Proof

We know $x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$

$$\mathcal{F}\,[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)h(n-k)e^{-j\omega n}$$

Interchanging the order of summation we get

$$\mathcal{F}\,[x_1(n) * x_2(n)] = \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k)e^{-j\omega n}$$

put $n - k = P$, then

$$\mathcal{F}\,[x_1(n) * x_2(n)] = \sum_{k=-\infty}^{\infty} x(k) \sum_{P=-\infty}^{\infty} h(P)e^{-j\omega P}e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x(k)H(e^{j\omega})e^{-j\omega k}$$

$$= H(e^{j\omega}) \sum_{K=-\infty}^{\infty} x(k)e^{-j\omega k}$$

$$= H(e^{j\omega})X(e^{j\omega})$$

	Differentiation in frequency If $\mathscr{F}[x(n)] = X(e^{j\omega})$ then $\mathscr{F}[nx(n)] = \frac{jd}{d\omega}X(e^{j\omega})$ (9.45) **Proof** $X(e^{j\omega}) = \mathscr{F}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$			
	Part C(20 Mark Questions with Key)			
1	**State and prove the properties of z-transform**	20	4	K2

Properties of Z-transform:

Some Important properties of the z-transform that make it a powerful tool for analysis and design of discrete LTI systems is discussed below.

Many of these properties are analogous to those of Fourier Transform.

1) **Linearity:** If $X_1(z) = Z\{x_1(n)\}$ and $X_2(z) = Z\{x_2(n)\}$,

then

$$Z\{a x_1(n) + b x_2(n)\} = a X_1(z) + b X_2(z) \rightarrow (1)$$

Proof:

$$Z\{a x_1(n) + b x_2(n)\} = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$Z\{a x_1(n) + b x_2(n)\} = a X_1(z) + b X_2(z)$$

2) **Time shift or translation:**

If $X(z) = Z\{x(n)\}$ and the initial conditions for $x(n)$ are zeros, then $Z\{x(n-m)\} = z^{-m} X(z) \rightarrow (2)$

where m is +ve or –ve integer.

Proof:

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

$$= z^{-m} \sum_{n=-\infty}^{\infty} x(n-m) z^{-(n-m)}$$

Let $(n-m) = l$, $= z^{-m} \sum_{l=-\infty}^{\infty} x(l) z^{-l}$

$$Z\{x(n-m)\} = z^{-m} X(z)$$

except at z=0 or z=

3) Multiplication by an exponential sequence :

If $X(z) = Z\{x(n)\}$ then, $Z\{a^n x(n)\} = X(a^{-1}z) \rightarrow (3)$

Proof:

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$$

where ROC is $|a| R_1 < |z| < |a| R_2$

$$= X(a^{-1}z)$$

4) Time Reversal :

If $X(z) = Z\{x(n)\}$, then $Z\{x(-n)\} = X(z^{-1}) \rightarrow (4)$

Proof:

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) z^{l} \quad \text{where } l = -n$$

$$= \sum_{l=-\infty}^{\infty} x(l) (z^{-1})^{-l}$$

$$= X(z^{-1})$$

ROC

$$\frac{1}{R_2} < |z| < \frac{1}{R_1}$$

5) Differentiation of $X(z)$:

If $X(z) = Z\{x(n)\}$, then $Z\{n x(n)\} = -z \frac{d}{dz} X(z) \rightarrow (5)$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Differentiating z-transform

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz}(z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} x(n) (-n) z^{-n-1}$$

$$\frac{d}{dz}X(z) = \frac{-1}{z}\sum_{n=-\infty}^{\infty} n\,x(n)\,z^{-n}$$

multiply on both sides $(-z)$, we get

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} n\,x(n)\,z^{-n}$$

$$= Z\{n\,x(n)\}$$

6) Convolution Theorem:

If $X(z) = Z\{x(n)\}$, $H(z) = Z\{h(n)\}$, then

$$Z\{x(n) * h(n)\} = X(z)\,H(z) \longrightarrow (6)$$

where $x(n) * h(n)$ denotes linear convolution of sequences.

Proof: we have $y(n) = x(n) * h(n)$

$$= \sum_{n=-\infty}^{\infty} x(k)\,h(n-k)$$

$$Y(z) = Z\{y(n)\} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x(k)\,h(n-k)\right] z^{-r}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x(k)\,z^{-k}\,h(n-k)\,z^{-(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\,z^{-k} \sum_{n=-\infty}^{\infty} h(n-k)\,z^{-(n-k)},$$

Replace $(n-k)$ by l

$$= \sum_{k=-\infty}^{\infty} x(k)\,z^{-k} \sum_{l=-\infty}^{\infty} h(l)\,z^{-l}$$

$$Y(z) = X(z)\,H(z)$$

7) Correlation:

If $X_1(z) = Z\{x_1(n)\}$ and $X_2(z) = Z\{x_2(n)\}$, then

$$Z\left[r_{x_1 x_2}(l)\right] = Z\left[\sum_{n=-\infty}^{\infty} x_1(n)\,x_2(n-l)\right] = \Gamma_{x_1 x_2}(z) = X_1(z)\,X_2(z^{-}$$

(8)

8) Complex Convolution theorem

The z-transform of the product of 2 sequences is related to the z-transforms of the individual sequences through the complex convolution theorem.

This theorem states that if $x_3(n) = x_1(n) x_2(n)$, then

$$X_3(z) = \frac{1}{2\pi j} \oint_C X_1(v) X_2\left(\frac{z}{v}\right) v^{-1} dv \rightarrow (8)$$

The convergence region for $X_3(z)$ consists of all z such that if v is in the region of convergence for $X_1(z)$, then z/v is in the region of convergence of $X_2(z)$.

9) Parseval's Relation : 2m

Consider 2 complex sequences $x_1(n)$ and $x_2(n)$.

Parseval's relation states that,

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv \rightarrow (9)$$

where, the contour of integration must be in the overlap of the regions of convergence of $X_1(v)$ and $X_2^*\left(\frac{1}{v^*}\right)$.

Proof: Let $y(n) = x_1(n) x_2^*(n)$

$$Y(z) = \sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) z^{-n}$$

using complex convolution theorem, we can write above eqn. as

$$Y(z) = \frac{1}{2\pi j} \oint_C X_1(v) X_2^*\left(\frac{z^*}{v^*}\right) v^{-1} dv$$

Evaluating for $z=1$ gives

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv$$

If $X_1(z)$ and $X_2(z)$ both converge on the unit circle, we can choose $v = e^{j\omega}$ and, above eqn becomes,

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\, d\omega$$

which is Parseval's relation in terms of Fourier transform.

$v^{-1}dv = e^{-j\omega} j e^{j\omega} d\omega = j\,d\omega$

$\frac{1}{v^*} = \frac{1}{e^{-j\omega}} = e^{j\omega}$

If $x_1(n) = x_2(n) = x(n)$, where $x(n)$ is a real sequence, then

$$\sum_{n=-\infty}^{\infty} x^2(n) = \frac{1}{2\pi j}\oint_C X(v)\, X(v^{-1})\, v^{-1}\, dv$$

where C is a closed contour in the ROC of $X(z)$.

10) Initial Value theorem:

If $X_+(z) = Z\{x(n)\}$, then $x(0) = \lim_{z\to\infty} X_+(z)$ ↳(10)

Proof: $X_+(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$, As $z \to \infty$, all the term vanish expect $x(0)$, which proves the theorem.

$$\therefore \lim_{z\to\infty} X_+(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x(n)\, z^{-n} = x(0)$$

11) Final Value Theorem:

If $X_+(z) = Z\{x(n)\}$, where ROC for $X_+(z)$ includes, but not necessarily confined to $|z|>1$ and

2	**State sampling and explain how the original signals can be reconstructed from sampled version?**	20	4	K2

Band-limited signals:

A Band-limited signal is one whose Fourier Transform is non-zero on only a finite interval of the frequency axis.

Specifically, there exists a positive number **B** such that **X(f)** is non-zero only in $f \in [-B, B]$. **B** is also called the Bandwidth of the signal

To start off, let us first make an observation about the class of Band-limited signals.

Lets consider a Band-limited signal **x(t)** having a Fourier Transform **X(f)**.
Let the interval for which **X(f)** is non-zero be **-B ≤ f ≤ B**.

Then, $x(t) = \int_{-B}^{B} X(f)\, e^{j2\pi f t}\, df$ converges.

The RHS of the above equation is differentiable with respect to **t** any number of times as the integral is performed on a bounded domain and the integrand is differentiable with respect to t. Further, in evaluating the derivative of the RHS, we can take $\frac{d}{dt}$ inside the integral.

$$\frac{dx(t)}{dt} = \int_{-B}^{B} (j2\pi f)\, X(f)\, e^{j2\pi f t}\, df$$

In general,

$$\frac{d^n x(t)}{dt^n} = \int_{-B}^{B} (j2\pi f)^n X(f)\, e^{j2\pi f t}\, df$$

This implies that band limited signals are **infinitely differentiable**, therefore, very **smooth** .

We now move on to see how a Band-limited signal can be reconstructed from its samples.

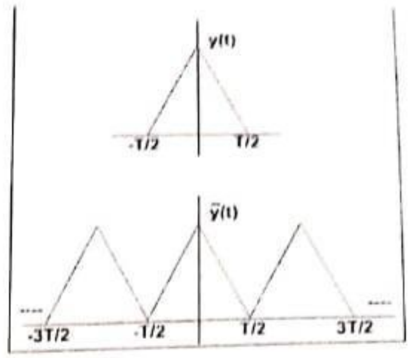
Reconstruction of Time-limited Signals

Consider first a signal **y(t)** that is **time-limited**, i.e. it is non-zero only in **[-T/2, T/2]**.

Its Fourier transform **Y(f)** is given by:

.

$$Y(f) = \int_{-T/2}^{T/2} y(t) e^{-j2\pi f t} dt = \int_{-\infty}^{+\infty} \tilde{y}(t) e^{-j2\pi f t} dt \quad \rightarrow (1)$$



Where $\tilde{y}(t)$ is the periodic extension of **y(t)** as shown

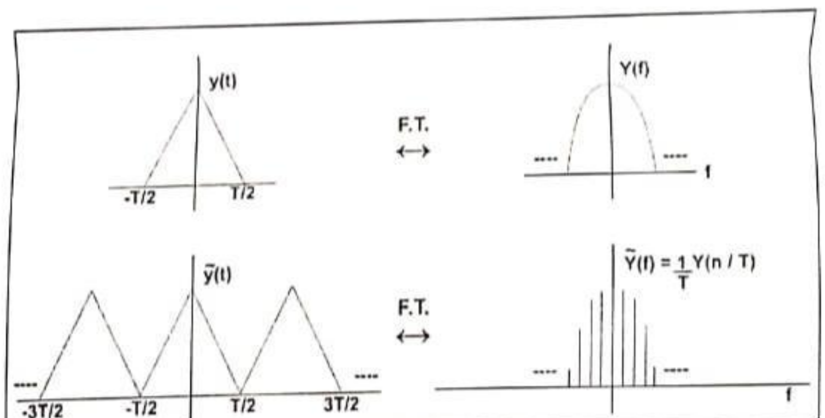
Now, Recall that the coefficients of the Fourier series for a periodic signal **x(t)** are given by :

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi n f_0 t} dt \quad \text{where} \quad f_0 = \frac{1}{T} \quad \text{—} \quad (2)$$

Comparing **(1)** and **(2)**, you will find

$$a_n = \frac{1}{T} Y\left(\frac{n}{T}\right)$$

That is, the Fourier Transform of the periodic signal $\tilde{y}(t)$ is nothing but the samples of the original transform.



Therefore, given that; **y(t) is time-limited in [-T/2, T/2]** and periodic, the entire information about **y(t)** is contained in jus **equispaced samples of its Fourier transform**! It is the dual of this result that is the basis of Sampling and Reconstruction of Band limited signals :-

Knowing the **Fourier transform is limited** to, say **[-B, B]**, the entire information about the transform (and hence the signal) is contained in just **uniform samples of the (time) signal** !

Reconstruction of Band-limited signals

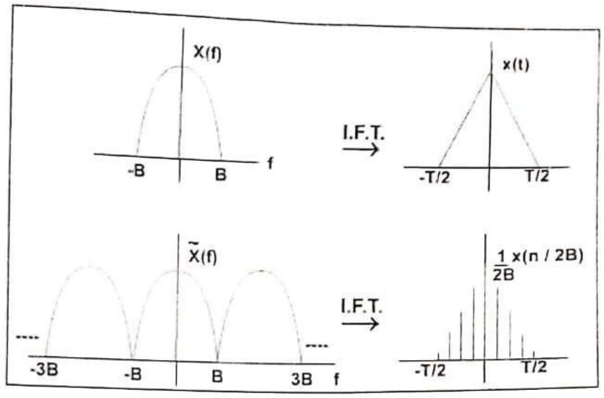
Let us now apply the dual reasoning of the previous discussion to Band-limited signals.

x(t) is Band-limited, with its Fourier transform **X(f)** being non-zero only in **[-B, B]**. The dual reasoning of the discussion in previou slide will imply that we can reconstruct **X(f)** perfectly in **[-B, B]** by using only the samples **x(n / 2B)**. Let's see how.

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi f t} df = \int_{-B}^{+B} X(f) e^{j2\pi f t} df$$

$$x\left(\frac{n}{2B}\right) = \int_{-B}^{B} X(f)\, e^{j2\pi \frac{n}{2B} f}\, df$$

This time, $\frac{1}{2B} x\left(\frac{n}{2B}\right)$ is the $-n^{th}$ Fourier series co-efficient of $\tilde{X}(f)$, the periodic extension of **X(f)**.



2B 2B 2B 2B 2B 2B

Thus we see that if we multiply the original Band-limited signal with a periodic train of impulses (period 1/2B, with impulse at the origin of strength 1/2B) we obtain a signal whose Fourier transform is a periodic extension of the original spectrum. So how does one retrieve the original signal from $\tilde{x}(t)$? We need a mechanism that will blank out the spectrum of $\tilde{x}(t)$ in $|f| > B$, i.e: multiply the spectrum with :

$$H(f) = \begin{cases} 1 & -B \le f \le B \\ 0 & \text{Otherwise} \end{cases}$$

- Band-limited signals are infinitely differentiable and very smooth.
- Given that 'x(t)' is **Band-limited** with its Fourier transform 'X(f)' being non-zero only in **[-B,B]** , we can say that $$\sum_{-\infty}^{\infty} \frac{1}{2B} x\left(\frac{n}{2B}\right) \delta\left(t-\frac{n}{2B}\right)$$ has a spectrum that is the **periodic extension** of 'X(f)' with period 2B.
- By passing $\sum_{-\infty}^{\infty} \frac{1}{2B} x\left(\frac{n}{2B}\right) \delta\left(t-\frac{n}{2B}\right)$ through an appropriate **Ideal Low-pass filter** one can obtain back 'x(t)'.